Energy Considerations in the Analysis of Stability of Non-conservative Structural Systems

by G. Herrmann and S. Nemat-Nasser

National Aeronautics and Space Administratic
Grant NsG 605

December 1965
Technical Report No. 65-5

ENERGY CONSIDERATIONS IN THE ANALYSIS OF STABILITY OF NONCONSERVATIVE STRUCTURAL SYSTEMS*

by

G. Herrmann
Professor of Civil Engineering
The Technological Institute
Northwestern University
Evanston, Illinois

and

S. Nemat-Nasser
Senior Research Engineer
Department of Civil Engineering
The Technological Institute
Northwestern University
Evanston, Illinois

* This research was supported by the National Aeronautics and Space Administration under Grant NsG 605.

ABSTRACT

24592

Energy considerations are introduced into the analysis of discrete, linear structural systems subjected to nonconservative (circulatory) forces. Various formulations of energy balance permit to establish stability criteria and to acquire further insight into several features associated with the stability of nonconservative systems, e.g. the destabilizing effect of damping and gyroscopic forces. A simple example is used to illustrate the general considerations.

1. Introduction

The investigation of stability of equilibrium of a linear mechanical system with a finite number of degrees of freedom under the action of a set of nonconservative forces has received considerable attention in recent years. In such systems the applied forces are said to be nonconservative in the sense that they do not possess a potential and they are assumed to be linear functions of the generalized coordinates. In addition, forces may be present in the system which are dependent on the generalized velocities and, therefore, may likewise be not derivable from a potential. The term "nonconservative" shall be reserved for coordinate-dependent forces which lack a potential (circulatory forces), while velocity-dependent forces will be designated as dissipative (or damping) and gyroscopic forces.

Ziegler, in two memoirs [1, 2]* has discussed various concepts of elastic stability and presented a comprehensive classification of systems and methods of analysis. In these studies Ziegler has shown, in particular, that although the usual equilibrium and energy methods yield conditions for the stability of a conservative system, these methods, in general, are inadequate for the study of a nonconservative system. Therefore, the dynamic method (or vibration method) must be employed for the analysis of the stability of such systems. It is for this reason that the problem of stability of equilibrium of nonconservative systems may be said to constitute a special branch of the broader area of problems concerned with dynamic stability of structures.

More recently, several investigators [3, 4, 5, 6, 7, 8, 9, 10, 11] have studied various nonconservative systems using the vibration method. For

---

* The numbers in brackets refer to the references listed at the end of the paper.

example in [3] a system with two degrees of freedom was employed to show that, when nonconservative forces are present, multiple regions of stability and instability may occur. It was also indicated that a nonconservative system may lose stability either by divergence (an adjacent equilibrium configuration exists), or by flutter (dynamic instability; oscillations with increasing amplitude). Further, from [3] one may conclude that the conditions for divergence can be obtained by employing either the static (equilibrium) or the dynamic method.

It may be readily recognized that the usual energy method (or "work method" [2]) becomes inapplicable in alalyzing the stability of equilibrium of a nonconservative system, regardless of whether stability is lost by divergence or by flutter. Since, however, energy considerations in conservative systems supply powerful tools for the study of stability, it appears to be desirable to carry them over to the analysis of nonconservative systems. In fact, it is somewhat surprising to observe that, to the authors' knowledge, only a single study was found which contained energy considerations. In investigating the dynamics of articulated pipes conveying fluid, T. B. Benjamin [12] invoked Hamilton's principle and discussed the energy transfer to the system.

It is the purpose of the present contribution to propose an extension to the usual energy method, such as to make it applicable for the stability analysis of nonconservative systems with and without velocity-dependent forces. In attempting to accomplish this extension it was found that the value of energy considerations, at least for linear systems, is different in conservative and nonconservative systems. Energy methods, as applied to conservative systems, supply approximate values of critical loads, whereas the main value of energy considerations in nonconservative systems, to which this paper is devoted, consists in supplying further insight into certain features of behavior typical of such systems. It may be possible, however, to make energy considerations form the foundation of approximate methods of analysis both in linear and nonlinear nonconservative systems. These possibilities will be discussed elsewhere.

2. Conservative Systems

We consider a holonomic, autonomous, linear system with r degrees of freedom, described by generalized normal coordinates q_n ; $n = 1,2,\ldots,r$. We assume that $q_n = 0$ define an equilibrium state and investigate its nature in the presence of a set of conservative generalized forces (stability problem). These forces are taken in the form

$$Q_n^{\ c} = F\, C_{nm}\, q_m \; ;$$
$$m,n = 1,2,\ldots,r, \tag{1}$$

where the summation convention is implied and will be employed in the sequel. The parameter F, $(0 \leq F < \infty)$, is the loading factor, and $C \equiv [C_{nm}]$ is a symmetric matrix.

For small motions of the system in the vicinity of its equilibrium position $q_n = \dot{q}_n = 0$ and at any time t, the kinetic energy is $T = \frac{1}{2}\, \dot{q}_n \dot{q}_n$, and the potential energy associated with the restoring forces may be written as $V = \frac{1}{2}\, \lambda_n^{\ 2}\, q_n^{\ 2}$, where λ_n are the natural frequencies of the load-free system (F=0) in the absence of any damping forces. (In the sequel we shall refer to V as the internal energy of the system.) The work of the conservative forces, $Q_n^{\ c}$, may be written as

$$W^c = \int Q_n^{\ c}\, \dot{q}_n dt = \frac{F}{2}\, C_{nm} q_n q_m \; . \tag{2}$$

We now wish to study this stability problem by applying the energy method which, as was shown by Pearson [13], is equivalent to the equilibrium method.

In formulating the energy method one states that the configuration $q_n = 0$ is stable if the internal energy of the system, V, is larger than the work done by the conservative forces, W^c, for any static deviation of the system from the equilibrium configuration. This, of course, leads to the requirement that the total potential energy of the system

$$\bar{V} = V - W^c \tag{3}$$

be a minimum at the configuration $q_n = 0$ for $F < F_{cr}$, where F_{cr} is the critical value of the load parameter. Thus it is seen that F_{cr} is the smallest value of F for which $\bar{V}$ is no longer positive definite. This critical value is obtained if we set det $|\lambda_n{}^2 - F_{cr} C_{nm}| = 0$, where det$|\ |$ denotes the determinant of the matrix [ ].

We consider now the presence of linear viscous damping forces of the form $\nu G_{nm}\dot{q}_m$, where $G \equiv [G_{nm}]$ is a symmetric non-negative matrix and ν is a magnitude parameter. Then for $F = F_{cr}$ the state $q_n = 0$ defines neutral equilibrium.

3. Nonconservative Systems

We now turn our attention to the case when in addition to the conservative forces Q_n^c, the system is subjected to a set of nonconservative forces (i.e. forces which are not derivable from a potential) of the form

$$Q_n^N = F\alpha \, \bar{C}_{nm} q_m \; ; \qquad m,n = 1,2,\ldots,r.$$

Here the parameter α, $(-\infty < \alpha < \infty)$, may be designated as the degree of nonconservativeness of these forces, and $\bar{C} \equiv [\bar{C}_{nm}]$ is a nonsymmetric matrix corresponding to the forces which are not derivable from a potential. The generalized forces now are

$$Q_n = Q_n^c + Q_n^N = F\,[C_{nm} + \alpha\, \bar{C}_{nm}]\, q_m \; ;$$
$$m,n = 1,2,\ldots,r. \tag{4}$$

The work of the applied forces is no longer given by equation (2) but instead by

$$W = W^c + W^N = \frac{F}{2} C_{nm} q_n q_m + F\alpha \int \bar{C}_{nm} q_m \dot{q}_n dt;$$
$$m,n = 1,2,\ldots,r, \tag{5}$$

which depends not only on the final configuration of the system, but also on the path followed between the state $q_n = \dot{q}_n = 0$ and the state at time t. Therefore, the usual energy method as described in the previous section is no longer applicable and one must explore the possibility of its extension.

As has been pointed out by several authors [3,4,6,7,8], a nonconservative system may lose stability by either divergent-type motion, or by flutter. In the case of divergence, it seems possible to make the static energy method applicable in the presence of nonconservative forces by means of a slight modification.

Let us assume that the system loses stability by divergence. In this case, by definition, for $F = F_{cr\lambda}$, the system admits a nontrivial equilibrium configuration, say $q_n = A_n$; $n = 1,2,\ldots,r$. Now we consider small deviations from this configuration specified by $\delta q_n = \delta A_n$. In these deviations and to the first order of approximation in δA_n, the work of the generalized forces is $\delta W = F_{cr} (C_{nm} + \alpha\bar{C}_{nm})A_m \delta A_n$; $m,n = 1,2,\ldots,r$. Therefore, for neutral equilibrium, this work must be equal to the change of the internal energy of the system $\delta V = \sum_{n=1}^{r} \lambda_n^2 A_n \delta A_n$. But all δA_n are arbitrary constants and we have

$$\det \left| \lambda_n^2 - F_{cr.} (C_{nm} + \alpha \bar{C}_{nm}) \right| = 0, \tag{6}$$

as the criterion for divergence.

We note here that the above argument is essentially the same as the well-known virtual work method which is equivalent to the usual equilibrium approach to the stability analysis.

In the case of flutter, however, the system loses stability by perfooming amplified oscillations. In this case one must use the dynamic method [1,2] for stability analysis. Even though, as will be shown in the sequel, it is possible to formulate a single criterion which covers both cases, for the case of loss of stability by flutter a more restricted criterion will be suggested, which is better suited for application in specific problems.

4. Stability Criterion

To study the stability of a linear, nonconservative system, we consider the following energy functional E, representing the total energy at time t

$$E = T + V - W^c - W^N + D. \tag{7}$$

Here T is the kinetic energy, V the potential energy of the restoring forces (internal energy), $W^c + W^N$ the work of the generalized forces, and D the energy dissipated by damping in the system; e.i.

$$D = \nu \int G_{nm}\dot{q}_n\dot{q}_m dt.$$

For the equilibrium position, $q_n = \dot{q}_n = 0$, to be stable the functional E must be positive-definite for all admissible paths in the 2r-dimensional phase-space, connecting the origin $q_n = \dot{q}_n = 0$ to a state at time t.

This requirement is in accord with Liapunov's stability theorem [14]. It is, in fact, an extension of the usual stability requirement in energy terms, as discussed in Section 2 for conservative systems. This may be easily seen if we let $\alpha = 0$, ($W^N = 0$), and consider only paths which are normal to all $\dot{q}_n$ coordinate axes in the phase-space. The functional E then becomes identical to $\overline{V}$ introduced in Section 2.

If the loss of stability occurs by flutter, however, the above stability requirement yields a new criterion which cannot be deduced by static considerations. Let us assume that $V - W^c$ is a positive-definite quantity. Then, as T is always a positive definite functional, E will obviously be positive definite if $D - W^N$ is positive definite for all admissible paths in the 2r-dimensional phase-space. Therefore, we have the following sufficiency theorem for stability with respect to flutter.

In order that the equilibrium position $q_n = \dot{q}_n = 0$ be stable with respect to flutter, it is sufficient that $D - W^N$ be a positive-definite functional for all admissible paths in the 2r-dimensional phase-space.

Although the above sufficiency condition for stability is quite rigorous, it is not suited for application to specific problems. Therefore, we would like to suggest a heuristic stability criterion which may be inferred from the above statement of sufficiency.

Let us assume that the loss of stability occurs by flutter-type motion. In this case, by definition, beyond the threshold of stability, the system performs oscillations with increasing amplitudes. Then, along the real path in phase-space, we have, by virtue of conservation of energy,

$$\Delta E = \Delta(T+V-W^C) + \int_{t_1}^{t_2} [\nu G_{nm}\dot{q}_m - F\alpha\,\overline{C}_{nm}q_m]\,\dot{q}_n dt = 0,$$

where Δ denotes the change of $(T+V-W^C)$ from t_1 to t_2 .
Therefore, the conservative part of ΔE, $\Delta E^C = \Delta(T+V-W^C)$, increases if

$$\overline{D} - \overline{W}^N = \int_{t_1}^{t_2} [\nu G_{nm}\dot{q}_m - F\alpha\,\overline{C}_{nm}q_m]\,\dot{q}_n dt$$

is negative, and decreases if $\overline{D} - \overline{W}^N$ is positive. At the threshold of stability, the system performs steady-state oscillations at a fixed frequency ω. Then, by setting $t_1 = 0$, and $t_2 = \frac{2\pi}{\omega}$, this threshold may be defined by the equation

$$\overline{D} - \overline{W}^N = 0,$$

which establishes the flutter criterion.

Let us note, in this connection, a most remarkable feature of damping forces [15,16,17,18,19,20]. They not only provide for energy dissipation and are thus associated with an energy sink, but also, by virtue of influencing the phase difference among the various degrees of freedom, control the magnitude of the work done by the nonconservative forces Q_n^N. This latter property is associated with the velocity-dependence of the damping forces. Indeed, it can be shown that in the case of gyroscopic forces this property is, in general, retained, whereas, by definition of these forces, no energy is dissipated. To use a somewhat different description, we may say that velocity-dependent forces may or may not dissipate energy, but that in a nonconservative system such forces, in general, control the width of the channel through which the nonconservative forces Q_n^N can do work on the system.

By contrast to the above case, it can be shown that, when no velocity-dependent forces exist in the system, the work of the nonconservative forces averages to zero, through each cycle of oscillations, for all values of the load parameter smaller than a certain critical value, F_{cr}. In this case, the system has r distinct modes of oscillation which may be excited independently by suitable initial disturbances.

On the other hand, for $F > F_{cr}$, $\overline{W}^N$ is, in general, not equal to zero and the conservative energy of the system increases after each cycle. The system may, however, possess at most r-2 distinct modes of oscillation which may be excited independently by suitable initial disturbances. But for all other initial perturbations, the motion of the system becomes dominated by an oscillatory motion with increasing amplitudes and at a single frequency ω_c. We can conclude that, when velocity-dependent forces are not present, the system may be placed into steady-state motion by certain specific initial perturbations for $F > F_{cr}$. For all other initial perturbations, however, it becomes unstable by oscillations with increasing amplitudes. That is, energy is supplied to the system by the work of nonconservative forces and the conservative part of the energy of the system increases without bounds as $t \to \infty$.

It may be of interest to note again that the existence of gyroscopic forces, which are associated with a skew-symmetric matrix $\beta \equiv [\beta_{nm}]$, $\beta_{nm} = -\beta_{mn}$, $\beta_{nn} = 0$, provides only the channel for the transfer of energy and, therefore, renders the system unstable for all $F > 0$. This was first noted in [21] for systems with two and three degrees of freedom. Here we can easily recognize this property in the light of the energy considerations for systems with r degrees of freedom. We replace the matrix $G \equiv [G_{nm}]$ by $\beta \equiv [\beta_{nm}]$, and obtain $\bar{D} = 0$. The term $\bar{W}^N$, however, is not zero due to the existence of phase differences among the various degrees of freedom which is the required condition for nonconservative forces to do work on the system for all $F > 0$.

We now consider the case when ν is sufficiently small so that terms associated with ν^2 may be neglected in comparison with those of $O(\nu)$. It then can be easily shown that $\bar{W}^N$ is proportional to ν; e.i. $\bar{W}^N = \nu \bar{W}_1{}^N$. The flutter criterion then becomes

$$\bar{D} - \bar{W}^N = \nu \left(\int_0^{\frac{2\pi}{\omega}} G_{nm} \dot{q}_n \dot{q}_m dt - \bar{W}_1{}^N \right) = 0,$$

which is independent of the magnitude of the damping and is highly influenced by the relative values of the damping coefficients G_{nm}.

Although we focussed our attention on the stability analysis of discrete, linear nonconservative systems, all our results can be extended to continuous systems as was done by the present authors in [22]. Moreover, the method may also be employed for an approximate flutter analysis of nonlinear, discrete and continuous systems.

A detailed discussion of nonlinear flutter analysis will be presented elsewhere. We conclude this study by considering in the following section a simple example which illustrates most of the results obtained in this section.

5. Example

We consider a system with two degrees of freedom subjected to a follower force P, as is sketched in Fig. 1. This system was studied in detail by several authors [15,16,17]. The generalized coordinates are taken to be φ_1 and φ_2, and the joints A and B are assumed to be viscoelastic such that the restoring moments are $c\varphi_1 + b_1 \frac{d\varphi_1}{dt}$, and $c(\varphi_2 - \varphi_1) + b_2\left(\frac{d\varphi_2}{dt} - \frac{d\varphi_1}{dt}\right)$, respectively. With the other parameters of the system defined in Fig. 1, the equations of motion are

$$3m\ell^2\ddot{\varphi}_1 + m\ell^2\ddot{\varphi}_2 + (b_1 + b_2)\dot{\varphi}_1 - b_2\varphi_2 + (2c - P\ell)\varphi_1 + (P\ell - c)\varphi_2 = 0$$

$$m\ell^2\ddot{\varphi}_1 + m\ell^2\ddot{\varphi}_2 - b_2\dot{\varphi}_1 + b_2\dot{\varphi}_2 - c\varphi_1 + c\varphi_2 = 0, \qquad \text{(a)}$$

where dots denote differentiation with respect to time t.

We assume a solution in the form $\varphi_k = z_k e^{i\omega t}$; $z_1 = 1$, $z_2 = z = z_r + iz_i$, and obtain from (a)

$$z = z_r + iz_i = \left[1 - 2\,\frac{m\ell^2\omega^2(m\ell^2\omega^2 - c)}{(m\ell^2\omega^2 - c)^2 + \omega^2 b_2^2}\right] - i\left[\frac{2m\ell^2\omega^3 b_2}{(m\ell^2\omega^2 - c)^2 + \omega^2 b_2^2}\right] \qquad \text{(b)}$$

where $i\omega$ is the purely imaginary root of the frequency equation, that is

$$\Omega^2 = \omega^2\left(\frac{m\ell^2}{c}\right) = \frac{B_1 + B_2}{B_1 + 6B_2},$$

where $B_k = \frac{b_k}{\ell(cm)^{1/2}}$; $k = 1,2$.

The work of the nonconservative components of the generalized forces, during the interval from $t = 0$ to $t = \frac{2\pi}{\omega}$, is

$$-\bar{W}^N = -\int_0^{\frac{2\pi}{\omega}} Q_k^N \dot{q}_k dt = +P\ell\pi z_i. \qquad \text{(c)}$$

Similarly, the energy dissipated in the system during this same interval may be written as

$$\overline{D} = \pi\omega \left[(b_1 + b_2) - 2b_2 z_r + |z|^2 b_2 \right] , \tag{d}$$

where $|z|^2 = z_r^{\,2} + z_i^{\,2}$. Setting $\overline{D} - \overline{W}^N = 0$ we obtain* for B_1, $B_2 > 0$,

$$F = \frac{4B_1^{\,2} + 33B_1B_2 + 4B_2^{\,2}}{2(B_1 + B_2)(B_1 + 6B_2)} + \frac{1}{2} B_1B_2 , \tag{e}$$

where $F = \frac{P\ell}{c}$. From this equation we observe that, for B_1 and B_2 finite, F increases as damping increases. That is, F can be made as large as we please by selecting B_1 and B_2 large enough.

We now consider the case of small damping; namely, we neglect second order terms in B_1 and B_2 and obtain

$$z_r = -\frac{\Omega^2 + 1}{\Omega^2 - 1}$$

$$z_i = -\frac{2\Omega^3 B_2}{(\Omega^2 - 1)^2}$$

$$|z|^2 = (z_r)^2 = \frac{(\Omega^2 + 1)^2}{(\Omega^2 - 1)^2} . \tag{f}$$

Then by setting $\overline{D} - \overline{W}^N = 0$, we finally get

$$\frac{\overline{D} - \overline{W}^N}{\pi c^2} = \frac{2B_2\Omega^3}{(\Omega^2 - 1)^2} \left\{ -F + \frac{1}{2\Omega^2} \left[(4 + \beta)\Omega^4 - 2\beta\Omega^2 + \beta \right] \right\} , \tag{g}$$

---

* The same result can be obtained using Routh-Hurwitz criteria. See [16].

where $\beta = \frac{B_1}{B_2}$. For $B_2 \neq 0$, $(\Omega \neq 1)$, steady state motion is possible only if

$$F_d = \frac{4\beta^2 + 33\beta + 4}{2(1+\beta)(6+\beta)} , \tag{h}$$

which can also be obtained from (e) by neglecting $B_1 B_2$.

Equation (g) exhibits all properties of the system. It distinctly points out the effect of the ratio of damping coefficients on the critical load, the effect of vanishing damping, the degenerate case of $\Omega = 1$; $(B_2 = 0,\ B_1 \neq 0)$, and finally the limiting case of no damping. For B_2 small but fixed, the dissipation of energy by damping is greater than the input of energy as long as $F < F_d$. The system, therefore, is asymptotically stable. The maximum value of F is obtained for $\beta = 11.07$ and is $F_{max} = 2.086$ which coincides with the critical load of the undamped system $(B_1 = B_2 = 0)$.

For $B_2 = 0$ and $B_1 \neq 0$, $\Omega = 1$ and equation (g) has a factor 0/0 which yields no information. This is precisely the degenerate case when one of the equations of motion becomes uncoupled. In this case F = 2 for all values of B_1 . However, when $B_1 = 0$ but $B_2 \neq 0$, we have

$$\bar{D} - \bar{W}^N = \frac{2c\pi\sqrt{6}\,B_2}{25 + 6B_2^{\,2}} \left[\frac{1}{3} - F \right]$$

which yields $F = \frac{1}{3}$ independently of the order of magnitude of B_2 . Let us note here that, as was proved in [18], for $B_1 = 0$, $B_2 \neq 0$, and $B_1 \neq 0$, $B_2 = 0$, the determinant of the damping matrix becomes zero and the critical load of the undamped system establishes an upper bound for that of the damped system for all values of non-zero damping coefficients.

In the case of vanishing damping both B_1 and B_2 approach zero and therefore, for Ω real, $\bar{D}$ and $\vec{W}^N$ become identically equal to zero. In this case the critical value of F is obtained when $-W^N$ ceases to be a bounded, positive-definite quantity, i.e. when ω becomes complex which gives F = 2.086.

We now consider the effect of small gyroscopic forces. The equations of motion then are

$$\ddot{\varphi}_1 + \ddot{\varphi}_2 + B\dot{\varphi}_1 - \varphi_1 + \varphi_2 = 0$$

$$3\ddot{\varphi}_1 + \ddot{\varphi}_2 - B\dot{\varphi}_2 + (2-F)\varphi_1 + (F-1)\varphi_2 = 0,$$

where $\dot{\varphi}_k = \frac{d\varphi_k}{d\tau}$, $\tau = t\sqrt{\frac{c}{m\ell^2}}$. From these equations and to the first order of approximation in B, we obtain the frequency equation as

$$2\Omega^4 - \Omega^2(7 - 2F) - i\Omega BF + 1 = 0. \tag{i}$$

We now assume $\Omega = \lambda + iB\mu$ and substitute in the equation (i) to obtain

$$4\lambda^2 = 7 - 2F \pm \sqrt{4F^2 - 28F + 41},$$

$$\mu = \pm \frac{F}{2\sqrt{4F^2 - 28F + 41}},$$

which indicates that the system is unstable for all non-zero values of F.

ACKNOWLEDGEMENT

The authors wish to express their gratitude to Dr. J. M. T. Thompson, Department of Civil Engineering, University College London, for his constructive criticism received during the preparation of this paper.

This work represents a phase of research carried out at Northwestern University under Grant NsG 605 supported by the National Aeronautics and Space Administration.

REFERNCES

[1] H. Ziegler, "Linear Elastic Stability," Zeitschrift für angewandte Mathematik und Physik, Vol. 4, 1953, pp. 89-121, 168-185.

[2] H. Ziegler, "On the Concept of Elastic Stability," Advances in Applied Mechanics, Vol. 4, edited by H. L. Dryden and T. von Karman, Academic Press, Inc., New York, N. Y., 1965, pp. 351-403.

[3] G. Herrmann and R. W. Bungay, "On the Stability of Elastic Systems Subjected to Nonconservative Forces," Journal of Applied Mechanics, Vol. 31, 1964, pp. 435-440.

[4] S. Nemat-Nasser and G. Herrmann, "Torsional Instability of Cantilevered Bars Subjected to Nonconservative Loading," Journal of Applied Mechanics, in press.

[5] V. V. Bolotin, Nonconservative Problems of Theory of Elastic Stability, Moscow, 1961. English translation published by Pergamon Press, Inc., New York, N. Y., 1963.

[6] L. Contri, "Della Trave Caricata di Punta da Forze di Direzione Dipendente dalla sua Deformazione," Giornale de Genio Civile, Rome, Italy, 1964, pp. 32-39.

[7] Z. Kordas and M. Życzkowski, "On the Loss of Stability of a Rod Under a Super-Tangential Force," Archiwum Mechaniki Stosowanej, Vol. 15, No. 1, 1963, pp. 7-31.

[8] Z. Kordas, "Stability of the Elastically Clamped Compressed Bar in the General Case of Behaviour of the Loading," Bulletin de l'Academie Polonaise des Sciences, Serie des sciences techniques, Vol. 11, No. 12, 1963, pp. 419-426.

[9] T. R. Beal, "Dynamic Stability of a Flexible Missile under Constant and Pulsating Thrusts," AIAA Journal, Vol. 3, No. 3, 1965, pp. 486-494.

[10] H. Leipholz, "Anwendung des Galerkinschen Verfahrens auf nichtkonservative Stabilitätsprobleme des elastischen Stabes," Zeitschrift für angewandte Mathematik und Physik, Vol. 13, 1962, pp. 359-372.

[11] H. Leipholz, "Über den Einfluss eines Parameters auf die Stabilitätskriterien von nichtkonservativen Problemen der Elastomechanik," Ingenieur-Archiv, Vol. 34, 1965, pp. 256-263.

[12] T. B. Benjamin, "Dynamics of a System of Articulated Pipes Conveying Fluid," Proc. Roy. Soc. A, Vol. 261, 1961, pp. 457-486 (Part I).

[13] C. E. Pearson, "General Theory of Elastic Stability," Quarterly of Applied Mathematics, Vol. 14, 1956, pp. 133-144.

[14] J. LaSalle and S. Lefshetz, Stability by Liapunov's Direct Method with Applications, New York, N. Y., Academic Press, 1961.

[15] H. Ziegler, "Die Stabilitätskriterien der Elastomechanik," Ingenieur-Archiv, Vol. 20, 1952, pp. 49-56.

[16] G. Herrmann and I. C. Jong, "On the Destabilizing Effect of Damping in Nonconservative Elastic Systems," Journal of Applied Mechanics, Vol. 32, No. 3, 1965, pp. 592-597.

[17] G. Herrmann and I. C. Jong, "On Nonconservative Stability Problems of Elastic Systems with Slight Damping," Journal of Applied Mechanics, in press.

[18] S. Nemat-Nasser and G. Herrmann, "Some General Considerations Concerning the Destabilizing Effect in Nonconservative Systems," Zeitschrift für angewandte Mathematik und Physik, in press.

[19] G. Herrmann and S. Nemat-Nasser, "Instability Modes of Cantilevered Bars Induced by Fluid Flow Through Attached Pipes," to be published.

[20] S. Nemat-Nasser, S. N. Prasad, and G. Herrmann, "Destabilizing Effect of Velocity-Dependent Forces in Nonconservative Continuous Systems," AIAA paper No. 66-102.

[21] O. Bottema, "On the Stability of the Equilibrium of a Linear Mechanical System," Zeitschrift für angewandte Mathematik und Physik, Vol. 6, 1955, pp. 97-103.

[22] S. Nemat-Nasser and G. Herrmann, "On the Stability of Equilibrium of Continuous Systems," Ingenieur-Archiv, in press.

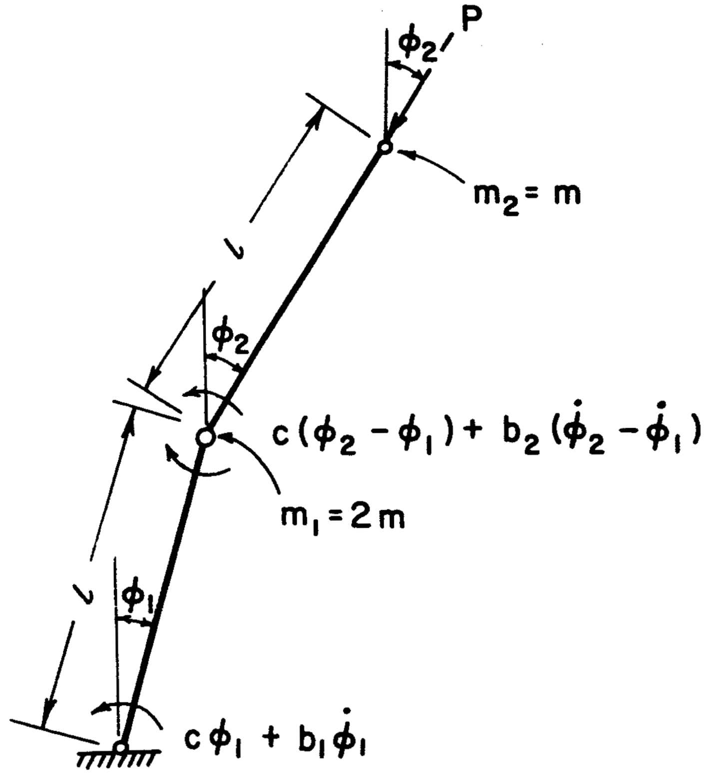
ϕ_2
P
$m_2 = m$
l
ϕ_2
$c(\phi_2 - \phi_1) + b_2(\dot{\phi}_2 - \dot{\phi}_1)$
$m_1 = 2m$
l
ϕ_1
$c\phi_1 + b_1\dot{\phi}_1$

Fig. 1